CALCULATION OF THE TEMPERATURE FIELD IN A PLANE CHANNEL WITH NONUNIFORM HEATING OF THERMALLY CONDUCTING WALLS

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A determination is made of the steady temperature of the walls of a channel with Poiseuille gas flow and with heat sources in the thermally conducting walls. The source power density has a parabolic distribution law.

A number of papers has recently appeared on the investigation of stream temperature profiles for Poiseuille flow in tubes and annular and plane channels, and under various heat transfer conditions [1–5]. In these no account is taken of heat due to internal friction and the work of pressure forces, nor of the heat flux along the wall.

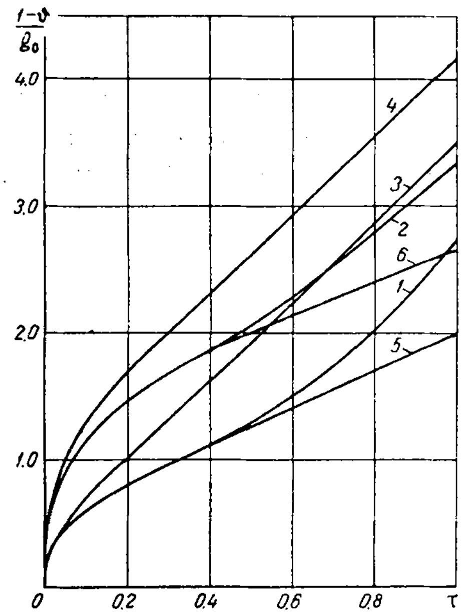

Fig. 1. Dependence of wall temperature on coordinate τ with inhomogenous heat sources, for $K = \infty$, $\vartheta_0 = 1$: 1) and 2) $g/g_0 = 1 + \tau^2$, $\varepsilon/g_0 = 0$ and 4, respectively; 3) and 4) $g/g_0 = 1 + 2\tau - \tau^2$, $\varepsilon/g_0 = 0$ and 4; 5) and 6) $g/g_0 = 1$, $\varepsilon/g_0 = 0$ and 4.

In the present paper, which is a continuation of [6], these factors are taken into account in calculating the temperature of the walls of a plane channel formed by two parallel, semi-infinite plane plates of thickness b, distant 2h apart. The thin walls of the channel contain heat sources whose power density depends on the coordinate x. The outside surfaces of the walls are thermally insulated, and the relative variation of absolute temperature in the channel is small. The temperature distribution at the channel inlet and the heating of the walls are symmetrical relative to the mid-plane of the channel.

The origin of coordinates Oxy is taken to be in the center of the inlet section of the channel, the Ox axis being directed along the flow, and Oy toward the wall.

The equation for the problem and part of the boundary conditions may be written as*

$$\rho c_p u \frac{\partial T}{\partial x} - \left[u \frac{dP}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2\right] = k \frac{\partial^2 T}{\partial y^2},$$

$$u = \frac{3}{2} u_m \left(1 - \left(\frac{y}{h}\right)^2\right), \quad u_m = -\frac{h^2}{3\mu} \frac{dP}{dx}, \tag{1}$$

$$\frac{\partial T}{\partial y} = 0 \text{ when } y = 0, \tag{2}$$

$$T = T_0(y) \text{ when } x = 0, \quad 0 \leqslant y < h. \tag{3}$$

The boundary condition at the wall (y = h) is obtained from the heat conduction equation for a thermally thin plate:

$$k_1 \frac{\partial^2 T}{\partial x^2} + G(x) - \frac{k}{b} \frac{\partial T}{\partial y} = 0 \text{ when } y = h. \tag{4}$$

Let the wall be thermally insulated at x = 0; then

$$\frac{\partial T}{\partial x} = 0 \text{ when } x = 0, \quad 0 \leqslant y \leqslant h. \tag{5}$$

We shall write (1)–(5) in dimensionless quantities:

$$(1 - \xi^2) \frac{\partial \vartheta(\tau, \xi)}{\partial \tau} - \varepsilon(1 - 3\xi^2) = \frac{\partial^2 \vartheta}{\partial \xi^2}, \tag{6}$$

$$\frac{\partial \vartheta}{\partial \xi} = 0 \text{ when } \xi = 0, \tag{7}$$

$$\vartheta = \vartheta_0(\xi) \text{ when } \tau = 0, \quad 0 \leqslant \xi < 1, \tag{8}$$

$$\frac{\partial^2 \vartheta}{\partial \tau^2} - K\left[g(\tau) + \frac{\partial \vartheta}{\partial \xi}\right] = 0 \text{ when } \xi = 1, \tag{9}$$

$$\frac{\partial \vartheta}{\partial \tau} = 0 \text{ when } 0 \leqslant \xi \leqslant 1. \tag{10}$$

When $K \gg 1$ we may neglect heat conduction along the wall. We note that the channel wall temperature at the

*It was shown in [2] that the thermal conductivity of the gas along the channel may be neglected when Re Pr > 10.

inlet $\vartheta(0,\ 1)$ is not equal to $\vartheta_0(1)$, but results from heat transfer along the channel walls and in the channel itself.

We carry out a Laplace transformation with respect to τ:

$$\frac{d^2\vartheta^*}{d\xi^2}=(1-\xi^2)\,p\vartheta^*-\vartheta_0(\xi)(1-\xi^2)-\frac{\varepsilon(1-3\xi^2)}{p} \tag{11}$$

with boundary conditions

$$\frac{d\vartheta^*}{d\xi}=0 \text{ when } \xi=0, \tag{12}$$

$$\frac{d\vartheta^*}{d\xi}=\frac{p^2\vartheta^*-p\vartheta(0,1)}{K}-g^* \text{ when } \xi=1. \tag{13}$$

The solution of (11), taking account of (12), may be written as [6]

$$\vartheta^*=\varphi\left[d+\int_0^{\xi}\varphi^{-2}\left(\int_0^{\xi'}\varphi h\,d\xi''\right)d\xi'\right], \tag{14}$$

$$\varphi(\xi)=\exp\left(-\frac{\sqrt{\alpha}}{2}\,\xi^2\right){}_1F_1\left(\frac{1-\sqrt{\alpha}}{4},\ \frac{1}{2},\ \sqrt{\alpha}\xi^2\right),$$

$$\alpha\equiv-p, \tag{15}$$

$$h=\vartheta_0(\xi)(\xi^2-1)+\frac{\varepsilon}{\alpha}(1-3\xi^2). \tag{16}$$

The function φ is the solution of the homogeneous equation

$$\frac{d^2\varphi}{d\xi^2}=(1-\xi^2)\,p\varphi. \tag{17}$$

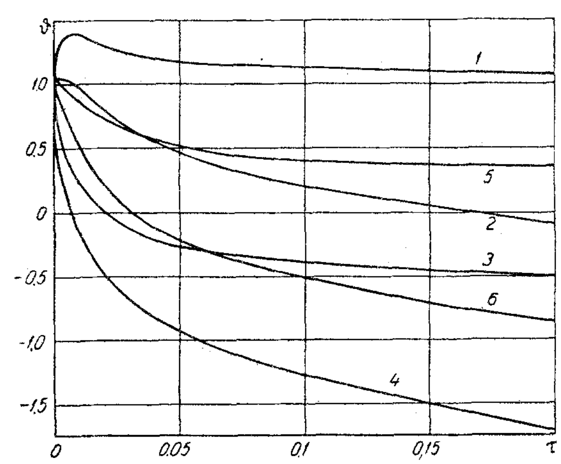

Fig. 2. Temperature distribution at the wall with $K=\infty$, $\varepsilon=2$ and nonuniform gas temperature at the inlet: 1) and 2) $\vartheta_0=2-\xi^2$, $g=0.5$ and 2, respectively; 3) and 4) $\vartheta_0=\xi^2$, $g=0.5$ and 2; 5) and 6) $\vartheta_0=1$, $g=0.5$ and 2.

We obtain the constant of integration d from condition (13):

$$d=\Delta^{-1}\left[-g^*+\alpha K^{-1}\vartheta(0,1)-\varphi^{-1}(1)\int_0^1\varphi h\,d\xi\right]+$$

$$+\Delta^{-1}\left[\left(\alpha^2K^{-1}\varphi(1)-\frac{\partial\varphi(1)}{\partial\xi}\right)\int_0^1\varphi^{-2}\int_0^{\xi}\varphi h\,d\xi'd\xi\right]. \tag{18}$$

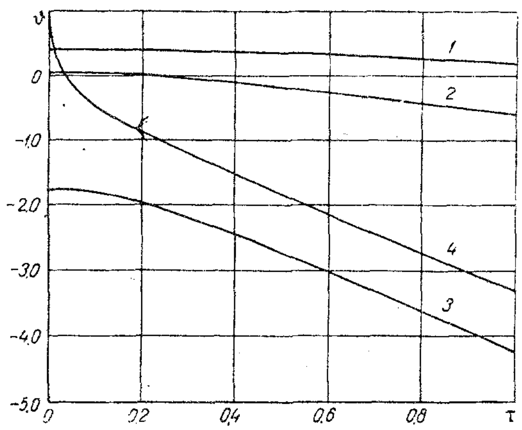

Fig. 3. Dependence of wall temperature ϑ on coordinate τ when there is heat flow along the wall ($\varepsilon=2$: 1), 2), 3) $g=0.2$; 0.5 and 2, respectively, $K=5$; 4) $g=2$; $K=\infty$.

Here

$$\Delta=\frac{\partial\varphi(1)}{\partial\xi}-\frac{\alpha^2\varphi(1)}{K}. \tag{19}$$

Putting $\xi=1$ in (14), we obtain an expression for the temperature at the wall:

$$\vartheta^*(1)=\Delta^{-1}\left[(\vartheta(0,1)\,\alpha K^{-1}-g^*)\,\varphi(1)-\int_0^1\varphi h\,d\xi\right]. \tag{20}$$

When $g^*=\varepsilon=0$ and $\vartheta_0(\xi)=1$ there is no heat transfer, and so $\vartheta(0,\ 1)=1$ and $\vartheta^*(1)=-\alpha^{-1}$. It follows from (16) and (20) that

$$-\Delta/\alpha=\alpha K^{-1}\varphi(1)-\int_0^1(\xi^2-1)\,\varphi\,d\xi. \tag{21}$$

We find the solution by dividing (20) by the right side of (21):

$$\vartheta^*(1)=-1/\alpha-\Delta^{-1}[g^*\varphi(1)-$$

$$(\vartheta(0,1)-1)\,\alpha K^{-1}\varphi(1)]-$$

$$-\Delta^{-1}\left[\varepsilon\alpha^{-1}\int_0^1(1-3\xi^2)\,\varphi\,d\xi+\int_0^1(\xi^2-1)(\vartheta_0-1)\,\varphi\,d\xi\right]. \tag{22}$$

For simplicity we shall restrict attention to the case of a parabolic dependence of heat source power density on coordinate τ:

$$g=g_0+g_1\tau+g_2\tau^2. \tag{23}$$

It follows that

$$g^*=g_0p^{-1}+g_2p^{-2}+2g_2p^{-3}. \tag{24}$$

It may be seen that φ is an integral function of α, and its expansion for small α is given by

$$\varphi(\xi, \alpha) = 1 - \frac{\alpha}{2}\xi^2 + \frac{\alpha}{12}\left(1 + \frac{\alpha}{2}\right)\xi^4 - \\ - \frac{\alpha^2(14+\alpha)}{720}\xi^6 + \frac{\alpha^2}{672}\xi^8 + \ldots \quad (25)$$

The poles of (22), taking (24) into account, and therefore p = 0 and the roots of the equation $\Delta = 0$. There are no other poles or branch points for expression (22).

We now write the complete solution for the wall temperature in the form of the sum

$$\vartheta(\tau) = \vartheta_0(\tau) + \vartheta_1(\tau), \quad (26)$$

where $\vartheta_0(\tau)$ is determined by the residues at the point p = 0 and gives the developed temperature realized at large τ. The function $\vartheta_1(\tau)$ is the sum of residues relative to the poles—the root of the equation $\Delta = 0$—and gives the wall temperature according to (26), along with $\vartheta_0(\tau)$.

To obtain the residue at the multiple pole p = 0 (i.e., to calculate the appropriate limits), we expand (22) in powers of p. Using (25), we obtain (22) in the form

$$\vartheta^*(1) = \frac{1}{p} + \frac{g_0 + g_1p^{-1} + 2g_2p^{-2} + 4\varepsilon p/35}{p^2[-2/3 + (b + K^{-1})p + cp^2 + dp^3]} - \\ - \frac{3}{2p}\int_0^1 (\xi^2 - 1)(\vartheta_0 - 1)\,d\xi. \quad (27)$$

Comparing (27) and (22), we find

$$-\frac{2}{3} + bp + pK^{-1} + cp^2 + dp^3 + \ldots = pK^{-1} + \\ + \frac{1}{\sqrt{\alpha}}\left[1 - (1+\sqrt{\alpha})\frac{{}_1F_1[(1-\sqrt{\alpha})/4,\ 3/2,\ \sqrt{\alpha}]}{{}_1F_1[(1-\sqrt{\alpha})/4,\ 1/2,\ \sqrt{\alpha}]}\right],$$

$$\alpha \equiv -p.$$

We expand the left side in a series with respect to $\sqrt{\alpha}$; the powers of α obtained cancel out, and b = = 0.21587, c = −0.075924, d = 0.026833.

Expansion (27) will be sufficient for calculating the residue of function $\vartheta^* \exp p\tau$ at the pole p = 0 of multiplicity 1–4. For the residues we have

$$\vartheta_0(\tau) = -\frac{6}{35}\varepsilon + \frac{3}{2}\int_0^1 (1-\xi^2)\vartheta_0(\xi)\,d\xi - \left(\frac{9}{4}b_1g_0 + \right. \\ \left. + \frac{9}{4}cg_1 + \frac{27}{8}g_1b_1^2 + \frac{9}{2}g_2d + \frac{27}{2}cb_1g_2 + \frac{81}{8}b_1^3g_2\right) - \\ - \left(\frac{3}{2}g_0 + \frac{9}{4}g_1b_1 + \frac{9}{2}g_2c + \frac{27}{4}g_2b_1^2\right)\tau - \\ - \left(\frac{3}{4}g_1 + \frac{9}{4}g_2b_1\right)\tau^2 - \frac{1}{2}g_2\tau^3, \quad b_1 = b + K^{-1}. \quad (28)$$

We determine the function $\vartheta_1(\tau)$. For this we find the residues of function (22) near the poles—the roots of the equation $\Delta = 0$:

$$\left(1 - \frac{\alpha\sqrt{\alpha}}{K}\right){}_1F_1\left(\beta, \frac{1}{2}, \sqrt{\alpha}\right) - \\ - (1+\sqrt{\alpha})\,{}_1F_1\left(\beta, \frac{3}{2}, \sqrt{\alpha}\right) = 0. \quad (29)$$

The roots α of (29) are real and positive. Determining the residues near the poles of α, we derive*

$$\vartheta_1(\tau) = \sum_\alpha \frac{2\exp(-\alpha\tau)}{\alpha\Phi_K} \times \\ \times \left\{(g_0 - g_1\alpha^{-1} + 2g_2\alpha^{-2})\,{}_1F_1\left(\beta, \frac{1}{2}, \sqrt{\alpha}\right) - \right. \\ - \varepsilon\exp\left(\frac{\sqrt{\alpha}}{2}\right)\int_0^1 \varphi(\xi, \alpha)(1 - 3\xi^2)\,d\xi - \\ - \alpha\exp\left(\frac{\sqrt{\alpha}}{2}\right)\int_0^1 (\xi^2 - 1)\vartheta_0\varphi\,d\xi + \\ \left. + \vartheta(0,1)\alpha^2K^{-1}\,{}_1F_1\left(\beta, \frac{1}{2}, \sqrt{\alpha}\right)\right\}. \quad (30)$$

Here

$$\Phi_K = \frac{3\alpha}{K}\,{}_1F_1\left(\beta, \frac{1}{2}, \sqrt{\alpha}\right) + {}_1F_1\left(\beta, \frac{3}{2}, \sqrt{\alpha}\right) - \\ - \left(1 - \frac{\alpha\sqrt{\alpha}}{K}\right)\sum_{j=1}^\infty\left\{\left[\beta(\beta+1)\ldots(\beta+j-1)\varphi_j\alpha^{j/2}\right] \times \right. \\ \left. \times \left[\frac{1}{2}\left(\frac{1}{2}+1\right)\ldots\left(\frac{1}{2}+j-1\right)j!\right]^{-1}\right\} + \\ + (1+\sqrt{\alpha})\sum_{j=1}^\infty\left\{[\beta(\beta+1)\ldots(\beta+j-1)\varphi_j\alpha^{j/2}] \times \right. \\ \left. \times \left[\frac{3}{2}\left(\frac{3}{2}+1\right)\ldots\left(\frac{3}{2}+j-1\right)j!\right]^{-1}\right\},$$

$$\varphi_j = \frac{j}{\sqrt{\alpha}} - \frac{1}{4}\sum_{l=1}^{j}\frac{1}{\beta + l - 1}. \quad (31)$$

When $K \to \infty$, function Φ_K becomes Φ, obtained in [6]. The sum of (28) and (30), according to (26), gives the solution of the problem. We find the unknown wall temperature at the channel inlet $\vartheta(0, 1)$ by putting τ = = 0 in the solution and solving the algebraic equation for $\vartheta(0, 1)$:

$$\vartheta(0,1) = M^{-1}\left[\frac{3}{2}\int_0^1 (1-\xi^2)\vartheta_0(\xi)\,d\xi - \frac{6}{35}\varepsilon - A_g\right] + \\ + M^{-1}\sum_\alpha \frac{2}{\alpha\Phi_K}\left[(g_0 - g_1\alpha^{-1} + 2g_2\alpha^{-2})\,{}_1F_1\left(\beta, \frac{1}{2}, \sqrt{\alpha}\right) - \right.$$

*Use was made in (31) of the relation, following from (21): $\frac{\alpha}{K}\,{}_1F_1\left(\beta, \frac{1}{2}, \sqrt{\alpha}\right) = \exp\left(\frac{\sqrt{\alpha}}{2}\right) \times \int_0^1 (\xi^2 - 1)\varphi(\xi, \alpha)\,d\xi$, where α are roots of (29).

$$-\varepsilon \exp\left(\frac{\sqrt{\alpha}}{2}\right)\int_0^1 (1-3\xi^2)\varphi\, d\xi -$$

$$-\alpha \exp\left(\frac{\sqrt{\alpha}}{2}\right)\int_0^1 (\xi^2-1)\varphi\vartheta_0\, d\xi\Big]. \quad (32)$$

The notation here is

$$M = 1 - \frac{2}{K}\sum_{\alpha}\frac{\alpha\, {}_1F_1(\beta,\ 1/2,\ \sqrt{\alpha})}{\Phi_K},$$

$$A_g = \frac{9}{4}b_1 g_0 + \frac{9}{4}c g_1 +$$

$$+\frac{27}{8}g_1 b_1^2 + \frac{9}{2}g_2 d + \frac{27}{2}c b_1 g_2 + \frac{81}{8}b_1^3 g_2. \quad (33)$$

A number of examples have been calculated according to (26), (28), and (30)–(33); the results are shown in Figs. 1–3. It may be seen that the dimensionless wall temperature is appreciably reduced (the true temperature increases) when longitudinal heat conduction in the wall is included. When $\tau = 0$ the temperature of the wall is different from that of the gas at the inlet ($\vartheta_0 = 1$).

NOTATION

c_p—specific heat; b—wall thickness; 2h—channel width; k—thermal conductivity of gas; k_1—the same, for the wall; P—pressure; $x \geq 0$—coordinate in stream direction; $0 \leq y \leq h$—coordinate from middle of channel to wall; G—heat source power density; T—absolute temperature; T_0—temperature at channel inlet; T_b—a fixed temperature, $T_b - T_0 \ll T_0$; u(y)—velocity of developed flow; u_m—mean velocity in channel; C—velocity of sound in gas; μ—viscosity; ρ—density. Dimensionless quantities: $\gamma = c_p/c_v$, $\varepsilon = \frac{9}{2}\frac{M^2(\gamma-1)\mathrm{Pr}}{\Theta_0}$, $\vartheta = \frac{T_b - T}{T_b - T_0}$, $\xi = \frac{y}{h}$, $\tau = \frac{x}{h}\frac{2}{3\mathrm{Pr\,Re}}$, $g = \frac{Gbh}{k(T_b - T_0)}$, $K = \frac{9}{4}\frac{h}{b}\frac{k}{k_1}(\mathrm{Pr\,Re})^2$, $M = u_m/C$, $\Theta_0 = (T_b - T_0)/T_0$, $\mathrm{Re} = u_m h\rho/\mu$, $\mathrm{Pr} = c_p\mu/k$; ${}_1F_1(a, b, x)$—degenerate hypergeometric function; $f^* = \int_0^1 f \exp(-p\tau)\, d\tau$; $\beta = (1-\sqrt{\alpha})/4$, $b_1 = b + K^{-1}$; p—variable in the transformed plane $\alpha \equiv -p$.

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